SECTION 4.9: ANTIDERIVATIVES

DEFINITION: A function F is an **antiderivative** of a function f on an interval I if $F'(x) = f(x)$ for all x in I.

EXAMPLE 1: Verify $F(x) = -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + 6$ is an antiderivative of $f(x) = x\sin(2x)$ for all x.

$$\begin{aligned}
F'(x) &= D_x\left[-\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + 6\right] \\
&= D_x\left[-\frac{1}{2}x\cos(2x)\right] + D_x\left[\frac{1}{4}\sin(2x)\right] + D_x[6] && \text{Sum and Difference Rule} \\
&= -\frac{1}{2}D_x\left[x\cos(2x)\right] + \frac{1}{4}D_x\left[\sin(2x)\right] + 0 && \text{Constant Multiple and Constant Rule} \\
&= -\frac{1}{2}\left(D_x[x]\cos(2x) + xD_x\left[\cos(2x)\right]\right) + \frac{1}{4}\cos(2x)D_x[2x] && \text{Product Rule and Chain Rule} \\
&= -\frac{1}{2}\left(\cos(2x) + x(-\sin(2x))D_x[2x]\right) + \frac{1}{4}\cos(2x)(2) && \text{Chain Rule} \\
&= -\frac{1}{2}\left(\cos(2x) - x\sin(2x)(2)\right) + \frac{1}{2}\cos(2x) \\
&= -\frac{1}{2}\cos(2x) + x\sin(2x) + \frac{1}{2}\cos(2x) \\
F'(x) &= x\sin(2x) \checkmark
\end{aligned}$$

THEOREM: Antiderivatives differ by at most a constant: If f is continuous on an interval I and F and G are antiderivatives of f on I then there is a constant C so that $F(x) = G(x) + C$.

PROOF: MVT is the MVP!

NOTATION: The notation $\int f(x)\,dx$ is called the **indefinite integral** of f.

$\int f(x)\,dx$ represents the **family** of antiderivatives of f.

EXAMPLE 2: Since $D_x\left[x^2\right] = 2x$, we write: $\int 2x\,dx = x^2 + C$.

EXAMPLE 3:

1. Use the fact $D_x[\sin(x)] = \cos(x)$ to find: $\int \cos(x)\,dx$.

$$\int \cos(x)\,dx = \sin(x) + C$$

2. Use the fact $D_x[\sin(2x)] = 2\cos(2x)$ to find: $\int \cos(2x)\,dx$.

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$$

PROPERTIES OF ANTIDERIVATIVES:

Suppose F and G are antiderivatives of f and g, respectively, on some open interval I.

- **CONSTANT MULTIPLE RULE:** $\displaystyle\int k \cdot f(x)\,dx = k\int f(x)\,dx = k \cdot F(x) + C$

- **SUM AND DIFFERENCE RULE:** $\displaystyle\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx = F(x) \pm G(x) + C$

- **BASIC FORMULAS:**

- $\displaystyle\int dx = \int 1\,dx = x + C$
- $\displaystyle\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C, \quad p \neq -1$
- $\displaystyle\int \sin(x)\,dx = -\cos(x) + C$
- $\displaystyle\int \cos(x)\,dx = \sin(x) + C$
- $\displaystyle\int \csc(x)\cot(x)\,dx = -\csc(x) + C$
- $\displaystyle\int \sec(x)\tan(x)\,dx = \sec(x) + C$
- $\displaystyle\int \csc^2(x)\,dx = -\cot(x) + C$
- $\displaystyle\int \sec^2(x)\,dx = \tan(x) + C$

OTHER COMMON FORMULAS:. If $k \neq 0$:

- $\displaystyle\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + C$
- $\displaystyle\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + C$
- $\displaystyle\int \csc(kx)\cot(kx)\,dx = -\frac{1}{k}\csc(kx) + C$
- $\displaystyle\int \sec(kx)\tan(kx)\,dx = \frac{1}{k}\sec(kx) + C$
- $\displaystyle\int \csc^2(kx)\,dx = -\frac{1}{k}\cot(kx) + C$
- $\displaystyle\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + C$

QUESTION: Why is there the restriction that $p \neq -1$ in the formula: $\displaystyle\int x^p\,dx = \frac{1}{p+1}x^{p+1} + C$?

EXAMPLE 4: Find the following antiderivatives. Check you answer by taking the derivative.

1. $\displaystyle\int \left(3x - \cos(2x) + \csc^2\left(\frac{x}{2}\right)\right) dx$

$$\begin{aligned}
\int \left(3x - \cos(2x) + \csc^2\left(\frac{x}{2}\right)\right) dx &= \int 3x\, dx - \int \cos(2x)\, dx + \int \csc^2\left(\frac{x}{2}\right) dx \\
&= 3\int x\, dx - \frac{1}{2}\sin(2x) - \frac{1}{\left(\frac{1}{2}\right)}\cot\left(\frac{x}{2}\right) + C \\
\int \left(3x - \cos(2x) + \csc^2\left(\frac{x}{2}\right)\right) dx &= \frac{3}{2}x^2 - \frac{1}{2}\sin(2x) - 2\cot\left(\frac{x}{2}\right) + C
\end{aligned}$$

To check, we verify $D_x\left[\frac{3}{2}x^2 - \frac{1}{2}\sin(2x) - 2\cot\left(\frac{x}{2}\right) + C\right] = \ldots = 3x - \cos(2x) + \csc^2\left(\frac{x}{2}\right) \checkmark$.

2. $\displaystyle\int \frac{\sqrt{x} - 3x^3}{4x^2}\, dx$

$$\begin{aligned}
\int \frac{\sqrt{x} - 3x^3}{4x^2}\, dx &= \int \left(\frac{\sqrt{x}}{4x^2} - \frac{3x^3}{4x^2}\right) dx \\
&= \int \frac{\sqrt{x}}{4x^2}\, dx - \int \frac{3x^3}{4x^2}\, dx \\
&= \frac{1}{4}\int \frac{\sqrt{x}}{x^2}\, dx - \frac{3}{4}\int \frac{x^3}{x^2}\, dx \\
&= \frac{1}{4}\int \frac{x^{\frac{1}{2}}}{x^2}\, dx - \frac{3}{4}\int x\, dx \\
&= \frac{1}{4}\int x^{-\frac{3}{2}}\, dx - \frac{3}{4}\left(\frac{1}{2}x^2\right) + C \\
&= \frac{1}{4}\frac{x^{-\frac{1}{2}}}{\left(-\frac{1}{2}\right)} - \frac{3}{8}x^2 + C \\
\int \frac{\sqrt{x} - 3x^3}{4x^2}\, dx &= -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{8}x^2 + C
\end{aligned}$$

Taking derivatives and doing some algebra, we verify $D_x\left[-\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{8}x^2 + C\right] = \ldots = \frac{\sqrt{x} - 3x^3}{4x^2} \checkmark$.

3. $\displaystyle\int \sec(x)[\sec(x) - \tan(x)]\, dx$

$$\begin{aligned}
\int \sec(x)[\sec(x) - \tan(x)]\, dx &= \int \left(\sec^2(x) - \sec(x)\tan(x)\right) dx \\
&= \int \sec^2(x)\, dx - \int \sec(x)\tan(x)\, dx \\
\int \sec(x)[\sec(x) - \tan(x)]\, dx &= \tan(x) - \sec(x) + C
\end{aligned}$$

To check, we find $D_x[\tan(x) - \sec(x) + C] = \ldots = \sec(x)[\sec(x) - \tan(x)] \checkmark$.

EXAMPLE 5: (VIDEO) Find the following antiderivatives and check your answers by taking the derivative.

1. $\displaystyle\int \frac{2-3x^2}{5\sqrt{x}}\,dx$

   Ans: $\displaystyle\int \frac{2-3x^2}{5\sqrt{x}}\,dx = \frac{4}{5}x^{\frac{1}{2}} - \frac{6}{25}x^{\frac{5}{2}} + C$

2. $\displaystyle\int [\cos(2\theta) - \sin(3\theta)]\,d\theta$

   Ans: $\displaystyle\int [\cos(2\theta) - \sin(3\theta)]\,d\theta = \frac{1}{2}\sin(2\theta) + \frac{1}{3}\cos(3\theta) + C$

3. $\displaystyle\int \frac{1}{\csc(4t)}\,dt$

   Ans: $\displaystyle\int \frac{1}{\csc(4t)}\,dt = \int \sin(4t)\,dt = -\frac{1}{4}\cos(4t) + C$, provided $\sin(4t) \neq 0$.

4. $\displaystyle\int \frac{1-\sin(x)}{\cos^2(x)}\,dx$

   Ans: $\displaystyle\int \frac{1-\sin(x)}{\cos^2(x)}\,dx = \int \left[\sec^2(x) - \sec(x)\tan(x)\right]dx = \tan(x) - \sec(x) + C$

EXAMPLE 6: Suppose the acceleration of an object moving along the line $x = 2$ is given by $a(t) = \sin(2t)$.

1. If the initial velocity of the object is -2, find a formula for the velocity of the object.

   Since $v'(t) = a(t)$, we have that $v(t) = \int a(t)\,dt$:

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int \sin(2t)\,dt \\ v(t) &= -\frac{1}{2}\cos(2t) + C \end{aligned}$$

   We are told the initial velocity of the object is -2,. In other words, $v(0) = -2$.
   Since $v(t) = -\frac{1}{2}\cos(2t) + C$, knowing $v(0) = -2$ allows us to solve for C:

$$v(0) = -2 \implies -\frac{1}{2}\cos(2(0)) + C = -2 \implies -\frac{1}{2} + C = -2 \implies C = -\frac{3}{2}$$

   Hence, $v(t) = -\dfrac{1}{2}\cos(2t) - \dfrac{3}{2}$.

2. If the initial position of the object is $(2,5)$, find a formula for the position of the object, $s(t)$.

   Since $v(t) = s'(t)$, we can find $s(t) = \int v(t)\,dt$:

$$\begin{aligned} s(t) &= \int v(t)\,dt \\ &= \int \left(-\frac{1}{2}\cos(2t) - \frac{3}{2}\right)\,dt \\ &= \int \left(-\frac{1}{2}\cos(2t)\right)\,dt - \int \left(\frac{3}{2}\right)\,dt \\ &= -\frac{1}{2}\int \cos(2t)\,dt - \frac{3}{2}\int\,dt \\ s(t) &= -\frac{1}{4}\sin(2t) - \frac{3}{2}t + C \end{aligned}$$

   Since the initial position if the object is $(2,5)$, we know $s(0) = 5$, hence:

$$s(0) = 5 \implies -\frac{1}{4}\sin(2(0)) - \frac{3}{2}(0) + C = 5 \implies C = 5$$

   Hence, $s(t) = -\frac{1}{4}\sin(2t) - \frac{3}{2}t + 5$.

EXAMPLE 7: (VIDEO)

A projectile launched directly upwards from a height h with initial velocity v_0 can be modeled by:

$$a(t) = -g, \qquad v(0) = v_0, \qquad s(0) = h.$$

Here, we assume the only force acting on the object is gravity and 'g' is the (constant) acceleration due to gravity.

Derive a formula for the height of the object off of the ground, $s(t)$ in terms of g, v_0, h, and t.

Ans: $s(t) = -\frac{1}{2}g\,t^2 + v_0\,t + h,\ t \geq 0.$

HOMEWORK: Section 4.9: 1 - 89 every other odd, 99*, 101 - 104 all.